1. (a) Use the binomial expansion, in ascending powers of x, to show that

$$\sqrt{(4-x)} = 2 - \frac{1}{4}x + kx^2 + \ldots$$

where k is a rational constant to be found.

(4)

A student attempts to substitute $x = 1$ into both sides of this equation to find an approximate value for $\sqrt{3}$.

(b) State, giving a reason, if the expansion is valid for this value of x.

(1)

a) $\sqrt{(4-x)} = (4-x)^{\frac{1}{2}}$ $-$ 1 $\quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$

$= 4^{\frac{1}{2}}(1-\frac{1}{4}x)^{\frac{1}{2}}$

$= 2(1-\frac{1}{4}x)^{\frac{1}{2}}$

$= 2\left(1 + \frac{1}{2}(-\frac{1}{4}x) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}(-\frac{1}{4}x)^2 + \ldots\right)$ $-$ 1

$= 2\left(1 - \frac{1}{8}x - \frac{1}{128}x^2 + \ldots\right)$ $-$ 1

$= 2 - \frac{1}{4}x - \frac{1}{64}x^2 + \ldots$

$\therefore k = -\frac{1}{64}$ $-$ 1

b) $(1+x)^n \Rightarrow$ valid for $|x| < 1$

$(1-\frac{1}{4}x)^{\frac{1}{2}} \Rightarrow$ valid for $|-\frac{1}{4}x| < 1$

$|-\frac{1}{4}x| < 1$

$|-x| < 4$

$|x| < 4$

The expansion is valid, as $x = 1$ fits into $|x| < 4$

2. (a) Use binomial expansions to show that $\sqrt{\frac{1+4x}{1-x}} \approx 1 + \frac{5}{2}x - \frac{5}{8}x^2$

(6)

A student substitutes $x = \frac{1}{2}$ into both sides of the approximation shown in part (a) in an attempt to find an approximation to $\sqrt{6}$

(b) Give a reason why the student **should not** use $x = \frac{1}{2}$

(1)

(c) Substitute $x = \frac{1}{11}$ into

$$\sqrt{\frac{1+4x}{1-x}} = 1 + \frac{5}{2}x - \frac{5}{8}x^2$$

to obtain an approximation to $\sqrt{6}$. Give your answer as a fraction in its simplest form.

(3)

Question continued

a)

$$\sqrt{\frac{a}{b}} = \sqrt{ab^{-1}} = a^{0.5}b^{-0.5}$$

$$\sqrt{\frac{1+4x}{1-x}} = \sqrt{(1+4x)(1-x)^{-1}} = (1+4x)^{0.5}(1-x)^{-0.5}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 \ldots$$
$n \in \mathbb{Q}$

$$(1+4x)^{0.5} = 1 + 0.5\times 4x + \frac{(0.5)(-0.5)}{2}\times 16x^2$$
$$= 1 + 2x - 2x^2$$

$$(1-x)^{-0.5} = 1 + 0.5x + \frac{(-0.5)(-1.5)}{2}\times x^2$$
$$= 1 + 0.5x + 0.375x^2$$

$$(1+4x)^{0.5}(1-x)^{-0.5} = (1+2x-2x^2)(1+0.5x+0.375x^2)$$
$$= 1 + 0.5x + 0.375x^2$$
$$2x + x^2$$
$$-2x^2$$
$$1 + 2.5x - 0.625x^2$$

$$\therefore \sqrt{\frac{1+4x}{1-x}} \approx 1 + \frac{5}{2}x - \frac{5}{8}x^2 \text{ as required.}$$

Question continued

b)

$$\sqrt{\frac{1+4x}{1-x}} \approx 1 + \frac{5}{2}x - \frac{5}{8}x^2$$

$(p+qx)^n$ valid when : $\left|\frac{qx}{p}\right| < 1$

$|qx| < |p|$

$|x| < \frac{|p|}{|q|}$

$1+4x \rightarrow \therefore |x| < \frac{1}{4}$ $\quad 1-x \rightarrow \therefore |x| < 1$

Expansion is valid when $|x| < \frac{1}{4}$

Since $x = \frac{1}{2} > \frac{1}{4}$, we shouldn't use $x = \frac{1}{2}$ ✓

Question continued

c)

@ $x = \frac{1}{11}$, $\sqrt{\frac{1+4x}{1-x}} = \sqrt{\frac{15/11}{10/11}} = \sqrt{\frac{15}{10}} = \sqrt{\frac{3}{2}}$

@ $x = \frac{1}{11}$, $1 + \frac{5}{2}x - \frac{5}{8}x^2 = 1 + \frac{5}{22} - \frac{5}{968}$

$= \frac{1183}{968}$ ✓

$\frac{\sqrt{3}}{\sqrt{2}} = \frac{1183}{968}$ ✓

$\sqrt{6} = \sqrt{3 \times 2} = \sqrt{3} \times \sqrt{2}$

$\frac{\sqrt{3}}{\sqrt{2}} \times 2 = \sqrt{6}$

$\frac{\sqrt{3}}{\sqrt{2}} \times \sqrt{2} = \sqrt{3}$

$\sqrt{3} \times \sqrt{2} = \sqrt{6}$

$\frac{\sqrt{3}}{\sqrt{2}} \times \sqrt{2} \times \sqrt{2}$

$= \sqrt{6}$

$\sqrt{6} = \frac{1183}{968} \times 2$

$= \frac{1183}{484}$

$\therefore \sqrt{6} = \frac{1183}{484}$ ✓

3. (a) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$(4-x)^{-\frac{1}{2}} \sim \frac{1}{\sqrt{4-x}}$$

$a^{-1} = \frac{1}{a}$ $\quad a^{\frac{1}{2}} = \sqrt{a}$

$a^{-\frac{1}{2}} = \frac{1}{\sqrt{a}}$

giving each coefficient in its simplest form.

(4)

The expansion can be used to find an approximation to $\sqrt{2}$
Possible values of x that could be substituted into this expansion are:

- $x = -14$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{18}} = \dfrac{\sqrt{2}}{6}$
- $x = 2$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $x = -\dfrac{1}{2}$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{\dfrac{9}{2}}} = \dfrac{\sqrt{2}}{3}$

(b) Without evaluating your expansion,

(i) state, giving a reason, which of the three values of x should not be used

(1)

(ii) state, giving a reason, which of the three values of x would lead to the most accurate approximation to $\sqrt{2}$

(1)

snippet from formula booklet

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times \ldots \times r}x^r + \ldots \quad (|x|<1,\ n\in\mathbb{R})}$$ ①

a) $(4-x)^{-\frac{1}{2}} = [(4-x)]^{-\frac{1}{2}} = \left[4\left(1-\frac{x}{4}\right)\right]^{-\frac{1}{2}}$

$= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$ ①

$\frac{1}{2}\left(1+-\frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\left[1 + \left(\frac{-1}{2}\right)\left(\frac{-x}{4}\right) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}\left(\frac{-x}{4}\right)^2 + \ldots\right]$ ①

$= \frac{1}{2}\left[1 + \frac{x}{8} + \frac{3x^2}{128} + \ldots\right]$

$\frac{1}{\sqrt{4-x}} = \frac{1}{2} + \frac{x}{16} + \frac{3x^2}{256} + \ldots$ ①

snippet from formula booklet

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times \ldots \times r}x^r + \ldots \quad (|x|<1, n\in\mathbb{R})$$

b) i) $\frac{1}{2}\left(1 + -\frac{x}{4}\right)^{-\frac{1}{2}}$ $\quad \left|\frac{-x}{4}\right| < 1 \quad |-x| < 4$

$\therefore |x| < 4$ (1)

$|-14| = 14 \quad 14 > 4$ which means $x = -14$ should not be used since $x = -14$ is not valid for $|x| < 4$

b) ii) $x = \frac{-1}{2}$ because it is closest to zero (1)

4. (a) Find the first four terms, in ascending powers of x, of the binomial expansion of

$$(1+8x)^{\frac{1}{2}}$$

giving each term in simplest form.

(3)

a) Find the first four terms of binomial expansion of $(1+8x)^{1/2}$

General Formula : $(1+y)^n = 1 + \frac{ny}{1!} + \frac{n(n-1)y^2}{2!} + \frac{n(n-1)(n-2)y^3}{3!} + \dots$ (four terms)

$(1+8x)^{1/2} \Rightarrow y = 8x$ and $n = \frac{1}{2}$ 1

$(1+8x)^{1/2} = 1 + \frac{\frac{1}{2}\times 8x}{1!} + \frac{\frac{1}{2}(\frac{1}{2}-1)(8x)^2}{2!} + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(8x)^3}{3!} + \dots$ 1

$(1+8x)^{1/2} = \underline{1 + 4x - 8x^2 + 32x^3} + \dots$ 1

(b) Explain how you could use $x = \frac{1}{32}$ in the expansion to find an approximation for $\sqrt{5}$

There is no need to carry out the calculation.

(2)

b) • We should substitute $x = \frac{1}{32}$ into $(1+8x)^{1/2}$ and this will give $\frac{\sqrt{5}}{2}$ 1

• We should then substitute $x = \frac{1}{32}$ into $1 + 4x - 8x^2 + 32x^3$ and we then multiply the result by 2 to give $\underline{\sqrt{5}}$. 1

5.

$$f(x) = \frac{50x^2 + 38x + 9}{(5x+2)^2(1-2x)} \qquad x \neq -\frac{2}{5} \quad x \neq \frac{1}{2}$$

Given that f(x) can be expressed in the form

$$\frac{A}{5x+2} + \frac{B}{(5x+2)^2} + \frac{C}{1-2x}$$

where A, B and C are constants

(a) (i) find the value of B and the value of C

(ii) show that $A = 0$

(4)

(b) (i) Use binomial expansions to show that, in ascending powers of x

$$f(x) = p + qx + rx^2 + \ldots$$

where p, q and r are simplified fractions to be found.

(ii) Find the range of values of x for which this expansion is valid.

(7)

(a) $\frac{50x^2+38x+9}{(5x+2)^2(1-2x)} = \frac{A}{5x+2} + \frac{B}{(5x+2)^2} + \frac{C}{1-2x}$

$50x^2 + 38x + 9 \equiv A(5x+2)(1-2x) + B(1-2x) + C(5x+2)^2$ 1

$x = 1/2$: $50(1/2)^2 + 38(1/2) + 9 = C(5(1/2)+2)^2$

$81/2 = 81/4\ C$

$C = 2$ 1

$x = -2/5$: $50(-2/5)^2 + 38(-2/5) + 9 = B(1-2(-2/5))$

$9/5 = 9/5\ B$

$B = 1$

$x = 0$: $9 = A(2)(1) + B(1) + C(2)^2$

$9 = 2A + B + 4C$

$= 2A + 1 + 4(2)$ 1

$= 2A + 9$

$A = 0$ 1

Question continued

$$\frac{50x^2+38x+9}{(5x+2)^2(1-2x)} \equiv \frac{1}{(5x+2)^2} + \frac{2}{(1-2x)} \quad *$$

(b)(i) $\frac{1}{(5x+2)^2} \equiv (5x+2)^{-2} \equiv \left[2\left(1+\frac{5}{2}x\right)\right]^{-2} \equiv 2^{-2}\left(1+\frac{5}{2}x\right)^{-2}$ (1)

$$\approx 2^{-2}\left\{1+\frac{(-2)\left(\frac{5}{2}x\right)}{1!}+\frac{(-2)(-3)\left(\frac{5}{2}x\right)^2}{2!}+\dots\right\} \quad (1)$$

$$= \frac{1}{4}\left\{1-5x+\frac{75}{4}x^2+\dots\right\}$$

$$= \frac{1}{4}-\frac{5}{4}x+\frac{75x^2}{16}+\dots \quad (1) \qquad |x| < \frac{2}{5}$$

$$\frac{2}{1-2x} \equiv 2(1-2x)^{-1}$$

$$\approx 2\left\{1+\frac{(-1)(-2x)}{1!}+\frac{(-1)(-2)(-2x)^2}{2!}+\dots\right\} \quad (1)$$

$$= 2\{1+2x+4x^2\}$$

$$= 2+4x+8x^2+\dots$$

$$\therefore f(x) = \frac{1}{4}-\frac{5x}{4}+\frac{75x^2}{16}+\dots+2+4x+8x^2+\dots \quad (1)$$

$$= \frac{9}{4}+\frac{11x}{4}+\frac{203x^2}{16}+\dots \quad (1) \qquad |x| < \frac{1}{2}$$

(ii) $|x| < \frac{2}{5}$ * (1) take the smaller value

$$\frac{2}{5} < \frac{1}{2}$$